MATH 590: QUIZ 6 SOLUTIONS

Name:

1. Calculate the determinant of $A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 0 \\ 1 & 0 & 3 \end{pmatrix}$ in two ways: First by expanding along the third column and second, by using elementary row operations. (5 points)

Solution. Expanding along the third column we have

$$|A| = 2 \cdot \begin{vmatrix} 0 & 2 \\ 1 & 0 \end{vmatrix} - 0 \cdot \begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix} + 3 \cdot \begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix} = 2(-2) + 0 + 6 = 2.$$

Using elementary row operations:

$$A \xrightarrow{-2 \cdot \mathrm{R}_1 + \mathrm{R}_3} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} = A'$$

Since this operation preserves the determinant and A' is upper triangular, $|A| = |A'| = 2$.

2. Find an orthonormal basis consisting of eigenvectors for the matrix $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$. Be sure to check that your basis is orthonormal. (5 points)

Solution. $\begin{vmatrix} x-1 & 2 \\ 2 & x-1 \end{vmatrix} = (x-1)^2 - 4 = x^2 - 2x - 3 = (x-3)(x+1)$, so the eigenvalues of A are $3, -1$.

For 3: Nullspace of $\begin{pmatrix} 1-3 & 2 \\ 2 & 1-3 \end{pmatrix}$ = nullspace of $\begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix}$ = nullspace of $\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$, which has basis $v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. This is a basis for the eigenspace of 3.

For -1: Nullspace of $\begin{pmatrix} 1-(-1) & 2 \\ 2 & 1-(-1) \end{pmatrix}$ = nullspace of $\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$ = nullspace of $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$, which has basis $v_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$. This is a basis for the eigenspace of -1.

Note that $v_1 \cdot v_1 = -1 + 1 = 0$, so these vectors are orthogonal, and hence any multiple of these vectors are orthogonal. Note further that v_1, v_2 have lengths $\sqrt{2}$, so if we set $u_1 := \frac{1}{\sqrt{2}} \cdot v_1$ and $u_2 := \frac{1}{\sqrt{2}} \cdot v_2$, then u_1, u_2 is an orthonormal basis consisting of eigenvectors for A.